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A Topological Interpretation for Vanishing of Higher Homotopy Groups of a Hypersolvable Arrangements

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Abstract:

In this paper we constructed a minimal CW complex for the complement of a supersolvable arrangement induced from the structure of its fundamental group as iterated semi direct product of finitely generated groups, in order to give a topological interpretation for vanishing of higher homotopy groups of the complement of a hypersolvable r-arrangement A, when we deformed it by Jambu's and Papadima's deformation method into a one parameter family of fiber-type arrangements $\{\tilde{A}_t\}_{t\in C}$.

Key words*:* Hyperplane arrangement, Supersolvable arrangement, fiber-type arrangement, hypersolvable arrangement, Orlike Solomon algebra, Cohomological ring, Higher homotopy groups, CW complex, Minimal CW complex.

Introduction:

The computation of the homotopy type of the complement to a hypersurface $Q^{-1}(0)$ is one of the essential problems in the topological study of the polynomial function $Q: C^{\ell} \to C$. We concerned with a polynomial Q which factors completely into distinct degree one factors. Then $Q = Q(A) = \prod_{i=1}^{n} \alpha_{H_i}$ is a defining polynomial of a complex hyperplane arrangements $A = \{H_1, \ldots, H_n\}$, where $\alpha_{H_i}: C^{\ell} \to C$ is a degree one factor of Q such that $\ker(\alpha_{H_i}) = H_i$, $1 \le i \le n$. The cohomology ring of the complement $M(A) = C^{\ell} \setminus \bigcup_{H\in A} H$ was computed by Brieskorn [5]. Many authores studied the fundamental group structure of $M(A)$ as generators and relations [3] and [18]. Much less is known about the higher homotopy groups.

For a fiber-type (supersolvable Stanley, [20]) arrangement all the higher homotopy groups of $M(A)$ are vanished and such arrangements are called $K(\pi = \pi_1(M(A)),1)$ arrangements [9]. The first computation of non trivial higher homotopy groups of $M(A)$ of a generic arrangement was made by Hattori [11]. Papadima and Suciu in [16], generalized Hattori's result to a hypersolvable arrangement and computed the first non vanishing higher homotopy group of $M(A)$. The hypersolvable class of arrangements was intoduced by Jambu and Papadima in (1998, [12]) and (2002, [13]), as a generalization of the fiber-type (supersolvable) class by using the collinear relations that encoded in the lattice intersection pattern up to codimension two $\mathrm{L}_2(A) = \{B \subseteq A \,|\, |B| \le 3\}$. They defined a vertical deformation $\{\tilde{A}_t\}_{t\in C}$ of a hypersolvable r-arrangement A which is not fiber-type such that for each $t \in C$, $\tilde{A}_t$ is fiber-type, A and $\tilde{A}_t$ have the same L_2 and they have isomorphic fundamental groups. Papadima and Suciu in [16] showed that the first non vanishing higher homotopy group of $M(A)$ has dimension;

$$p(M(A)) = sup\{k \mid P(H^*(M(A),Z),s) \equiv_{mod\, j} P(H^*(M(\tilde{A}_t),Z),s), \forall j \le k\};$$

where $P(H^*(M(A),Z),s)$ and $P(H^*(M(\tilde{A}_t),Z),s)$ are the Poincaré polynomials of the cohomological groups $H^*(M(A),Z)$ and $H^*(M(\tilde{A}_t),Z)$ respectively. Ali in [1], showed a conjecture of $p(M(A))$ as $p(M(A)) = p(A) = \max\{k \mid NBC_k(A) = S_k\}$ of $M(A)$. Randell in [17] showed that $M(A)$ of any complex hyperplane arrangement has homotopy type of a minimal (finite type) CW-complex, i.e. the number of the k-cells is equal to the k^{th}-Betti number $b_k(M(A)) = rk(H^k(M(A),Z))$. We refer the redear to [7], [8], [17], [22] and the references given there, as a recent work intrested with the minimal CW-complex of $M(A)$. However, little is known about the attaching map.

This paper is devoted to introduce a topological interpretation for an essential question; why, when we deformed a hypersolvable arrangement A which is not fiber type into a fiber type arrangement $\tilde{A}_t$, all the non-vanishing higher homotopy groups of the complement $\pi_n(M(A)),\ n \geq p(A)$ are vanished? Motivated by our purpose, we review some basic definitions. In section one, we introduce the notions of a "hypersolvable partition" of a complex arrangement A and a "hypersolvable ordering" on the hyperplans of A. Then we review a brief summary of a study given in [1] of the notion of "NBC bases" of a hypersolvable arrangement. Section two is devoted to mirror the properties of the hypersolvable partition on the structure of the cohomological ring of $M(A)$ and compute its Betti numbers which are the number of cells of the (finite type) minimal CW complex of the complement $M(A)$. In section three, we drive construction (3.1) of a minimal CW complex of a fiber type (supersolvable) arrangement and this construction was motivated by Switzer [21]. Finally, in section four our main result (theorem 4.2) is ststed and proved.

For the convenience of the reader we repeat the relevant material from [1] and [2] without proofs, thus making our exposition self contained.

1-A hypersolvable partition and the NBC bases of a hypersolvable arrangement

Let A be a central r-arrangement of hyperplanes over C. Define the complement $M(A) = C^r \setminus \bigcup_{i=1}^{n} H_i$ and $L(A)$ to be the lattice intersections of the hyperplanes of A reserves by inclusion, (i.e. $X \leq Y \Leftrightarrow Y \subseteq X$) and ranked by $rk(X) = co\dim(X)$. We refer the reader to [15] as a best general refrence.

Definition 1.1 [1] A partition $\Pi = (\Pi_1,...,\Pi_\ell)$ of A is said to be independent if the resulting ℓ-hyperplanes $H_j \in \Pi_j$, $1 \leq j \leq \ell$ are independent. Call $\mathrm{S} = \{H_{i_1},...,H_{i_k}\}$ a k-section of Π if for each $1 \leq j \leq k$, $H_{i_j} \in \Pi_{m_j}$ for some $1 \leq m_1 < \cdots < m_k \leq \ell$. Notice that if Π is independent, then all it's k-sections are independent. By $\mathrm{S}_\Pi^k(A)$ we denote the set of all k-sections of Π and $\mathrm{S}_\Pi(A) = \bigcup_{k=1}^{\ell} \mathrm{S}_\Pi^k(A)$. We call Π nice if it is independent and for each $X \in L_k(\mathrm{A})$, the induced partition $\Pi_X = (\Pi_X^1,...,\Pi_X^k)$ of $A_X = \{H \in A \mid X \subseteq H\}$, contains a singleton block, where for $1 \leq j \leq k$, $\Pi_X^j = \Pi_m \bigcap A_X \neq \phi$ for some $1 \leq m \leq \ell$.

Definition 1.2 [1] A partition $\Pi = (\Pi_1,...,\Pi_\ell)$ of A is said to be hypersolvable with length $\ell(A) = \ell$ and denoted by Hp, if $|\Pi_1| = 1$ and for fixed $2 \leq j \leq \ell$, the block Π_j satisfy the following properties:

(j^{th}-closed property of Π): For $H_1, H_2 \in \Pi_1 \bigcup \cdots \bigcup \Pi_j$, there is no $H \in \Pi_{j+1} \bigcup \cdots \bigcup \Pi_\ell$ such that $rk(H_1,H_2,H) = 2$.

(j^{th}-complete property of Π): For $H_1, H_2 \in \Pi_j$, there is $H \in \Pi_1 \bigcup \cdots \bigcup \Pi_{j-1}$ such that

$rk(H_1, H_2, H) = 2$. From $(j-1)^{\text{th}}$-closed property of Π, the hyperplane H must be unique we denoted by $H_{1,2}$.

(j^{th}-solvable property of Π): For $H_1, H_2, H_3 \in \Pi_j$, either $H_{1,2}, H_{1,3}, H_{2,3} \in \Pi_1 \cup \cdots \cup \Pi_{j-1}$ are equal or $rk(H_{1,2}, H_{1,3}, H_{2,3}) = 2$.

For $1 \le j \le \ell$, let $d_j = |\Pi_j|$. The vector of integers $d = (d_1, ..., d_\ell)$ is called the d-vector of Π and we define the rank of the blocks of Π as $rk(\Pi_j) = rk(\bigcap_{H \in \Pi_1 \cup \cdots \cup \Pi_j} H)$. We call Π_j singular if $rk(\Pi_j) = rk(\Pi_{j-1})$ and we call it non singular otherwise. We call a Hp Π is supersolvable if it is independent. Observe that $rk(\Pi_{j-1}) \le rk(\Pi_j)$ in general and if $\ell \ge 3$, then every $\Pi_{i_1}, \Pi_{i_2}, \Pi_{i_3} \in \Pi$ are independent.

Theorem 1.1 [2] A is hypersolvable if it has a hypersolvable partition and A is supersolvable if it has a supersolvable partition.

Definition 1.3 [1] Let A be a hypersolvable arrangement with Hp $\Pi = (\Pi_1, ..., \Pi_\ell)$. For $1 \le j \le \ell$, partitioned Π_j into two blocks as; put $\Pi_{j*1} = \{H_{i_1}, ..., H_{i_k}\} \subseteq \Pi_j$ such that $rk(H_{i_1}, ..., H_{i_k}) = 2$ and $\Pi_{j*2} = \Pi_j \setminus \Pi_{j*1}$. Define the induced hypersolvable ordering $\trianglelefteq$ of A as:

1. If $H \in \Pi_i$ and $H' \in \Pi_j$ such that $i < j$, put $H \trianglelefteq H'$. If $H \in \Pi_i$ and $H' \in \Pi_j$ such that $i < j$, put $H \trianglelefteq H'$.
2. For $2 \le j \le \ell$, give the hyperplanes of Π_{j*1} an arbitrary ordering such that if $H_1, H_2, H_3 \in \Pi_j$ with $rk(H_1, H_2, H_3) = 3$, put $H_{i_1} \trianglelefteq H_{i_2} \trianglelefteq H_{i_3}$ if, and only if, $H_{i_1,i_2} \trianglelefteq H_{i_1,i_3} \trianglelefteq H_{i_2,i_3}$, where $\{H_{i_1}, H_{i_2}, H_{i_3}\} = \{H_1, H_2, H_3\}$.

Definition 1.4 [15] A circuit $C \subseteq A$ is a minimal dependent subarrangement and for a given total order $\trianglelefteq$ of the hyperplanes of A, $\overline{\mathrm{X}} = \mathrm{X} \setminus \{H\}$ is said to be a broken circuit of A, where H is the smallest hyperplane of X with respect $\trianglelefteq$. The subarrangement $B \subseteq A$ which contains no broken circuit is denoted by NBC base of A and if $rkB=k$ we denoted it by k-NBC base. By $NBC_k(A)$ we denoted the set of all k-NBC bases of A and $NBC(A) = \bigcup_{k=1}^{r} NBC_k(A)$. Where the set of all k-broken circuits of A we denoted it by $BC_k(A)$ and $BC(A) = \bigcup_{k=1}^{r} BC_k(A)$.

Theorem 1.2 [2] Let A be a hypersolvable r-arrangement with Hp $\Pi = (\Pi_1, ..., \Pi_\ell)$. Then the following assertions are equivalent:

1. A is supersolvable. 2. Π is nice. 3. $NBC(A) = \mathrm{S}(A)$.

Definition 1.5 [16] Given two r-arrangements $A_1 = \{H_1^1, \ldots, H_n^1\}$ and $A_2 = \{H_1^2, \ldots, H_n^2\}$:

1. We will say A_1 and A_2 have the same lattice or L-equivalent and denoted by $L(A_1) \approx L(A_2)$, if for each $1 \le i_1 < \cdots < i_k \le n$ and $1 \le k \le n$ we have $rk(H_{i_1}^1, \ldots, H_{i_k}^1) = rk(H_{i_1}^2, \ldots, H_{i_k}^2)$.
2. For $2 \le k \le r-1$, set $\mathrm{L}_k(A_i) = \{B_i \subseteq A_i \mid |B_i| \le k+1\}$ to be the lattice intersection pattern up to codimension k of A_i

and $i=1,2$. We say A_1 and A_2 are L_k-equivalent and denoted by $\mathrm{L}_k(A_1)\approx\mathrm{L}_k(A_2)$ if for each $1\le i_1<\cdots<i_j\le n$ and $j\le k+1$ we have $rk(H^1_{i_1},\ldots,H^1_{i_J})=rk(H^2_{i_1},\ldots,H^2_{i_J})$.

Notice that, if A_1 and A_2 are L-equivalent, then they are L_k-equivalent for $2\le k\le r-1$. But the converse need not to be true in general.

Theorem 1.3 [1] let A_1 and A_2 be supersolvable r-arrangements. A_1 and A_2 are L-equivalent if and only if, they are L_2-equivalent.

Corollary 1.1 [2] Let A be a hypersolvable r-arrangement with Hp $\Pi=(\Pi_1,\ldots,\Pi_\ell)$ such that $r<\ell$, i.e. A is not supersolvable. Define $p(A)=\max\{k|\ NBC_k(A)=\mathrm{S}_k\}$. If $q=p(A)+1$ and $\mathrm{S}\in\mathrm{S}_q$ such that $\mathrm{S}\notin NBC_q(A)$, then S is a q-broken circuit via a hypersolvable ordering on the hyperplanes of A. That is, there exists $\mathrm{S}'\in\mathrm{S}_{p(A)+2}$ and S' is a $(p(A)+2)$-circuit.

Theorem 1.4 [1] Let A be a hypersolvable r-arrangement with Hp $\Pi=(\Pi_1,\ldots,\Pi_\ell)$ such that $r<\ell$ and let $\{\tilde{A}_t\}_{t\in C}$ be the Jambu's-Papadima's vertical deformation of A. Then, via a hypersolvable ordering on the hyperplanes of A:

1. $NBC(A)\subseteq\mathrm{S}$ and $NBC_2(A)=\mathrm{S}_2$.

3. Recall the definition of $p(A)$ in corollary (1.1). Then $2\le p(A)\le r-1$ such that for each $2\le k\le p(A)$, $NBC_k(A)=S_k$ and $NBC_{p(A)+1}(A)=\mathrm{S}_{p(A)+1}\setminus\mathrm{S}_{p(A)+1}\bigcap BC_{p(A)+1}(A)$.

4. For all $t_1,t_2\in C\setminus\{0\}$, $\tilde{A}_{t_1}$ and $\tilde{A}_{t_2}$ are L-equivalent.

5. For all $t\in C\setminus\{0\}$, A and $\tilde{A}_t$ are $\mathrm{L}_{p(A)}$-equivalent and Jambu's-Papadima's deformation destroyed all the dependent sections among any $(p(A)+2)$-different blocks of Π and replaced it by independent sections which adds a new NBC bases of $\tilde{A}(t)$.

6. If $r=3$, then $p(A)=2$.

2 *The hypersolvable partition complex, the Orlik-Solomon algebra and the cohomological rings of a hypersolvable arrangement*

The Orlik-Solomon algebra. ([14] and [15]) For any commutative ring K and an arbitrary total order $\trianglelefteq$ of A, defined the Orlik-Solomon algebra $\mathrm{A}^*_K(A)$ to be the quotient of the exterior K-algebra $E^*_K=\bigwedge^{j\ge0}(\bigoplus_{H\in\mathrm{A}}Ke_H)$, (see [10]), by a homogeneous ideal $I(\mathrm{A})$ generated by the relations $\sum_{j=1}^{k}(-1)^{j-1}e_{H_{i_1}}\ldots\hat{e}_{H_{i_j}}\ldots e_{H_{i_k}}$, for all $1\le i_1<\ldots<i_k\le n$ such that $rk(H_{i_1},\ldots,H_{i_k})<j$. The differentiation $\partial^*_E:E^*_K\to E^*_K$ is defined on E^*_K as; $\partial^0_E(e_{\{\}})=0$, $\partial^1_E(e_H)=1$ and for $2\le k\le n$;

$$\partial^k_E(e_S)=\sum_{j=1}^{k}(-1)^{j-1}e_{H_{i_1}}\ldots\hat{e}_{H_{i_j}}\ldots e_{H_{i_k}}\ ;$$

for each $S=\{H_{i_1},\ldots,H_{i_k}\}\subseteq A$. Notice that, $\partial^*_{\mathrm{A}}{:}\mathrm{A}^*_K(A)\to\mathrm{A}^*_K(A)$ which is defined by $\partial^*_{\mathrm{A}}=\psi^*\circ\partial^*_E$ inherits to $(\mathrm{A}^*_K(A),\partial^*_{\mathrm{A}})$ a structure of an a cyclic chain complex, where $\psi:E^*_K\to\mathrm{A}^*_K(A)$ is the canonical projection.

The partition complex [15] Let $\Pi=(\Pi_1,\ldots,\Pi_\ell)$ be a partition of A. A graded partition K-

module is defined to be $(\Pi)^*_K = (\Pi_1)^*_K \otimes \ldots \otimes (\Pi_\ell)^*_K$, where for $1 \le k \le \ell$, $(\Pi_k)^*_K$ is the free K-module with basis 1 and the elements of Π_k. For each $S \in \mathbf{S}_k$, define $q_S = x_1 \otimes \ldots \otimes x_\ell \in (\Pi)^*_K$ such that;

$$x_j = \begin{cases} H_{i_j} & \text{if } j = m_l, 1 \le l \le k; \\ 1 & \text{if } j \ne m_l, 1 \le l \le k. \end{cases}$$

Observe that $q_{\{\}} = 1$ and q_{S} is homogeneous of degree k. Denoted by $(\Pi)^k_K$ the k-homogeneous part of $(\Pi)^*_K$, then

$$(\Pi)^*_K = \bigoplus_{k=1}^{\ell} (\Pi)^k_K = \bigoplus_{k=1}^{\ell} \Big(\bigoplus_{\mathrm{S} \in \mathbf{S}_k} Kq_{\mathrm{S}} \Big).$$

Therefore, the graded K-module $(\Pi)^*_K$ has a basis is $\{q_{\mathrm{S}} \mid \mathrm{S} \in \mathbf{S}\}$.

Theorem 2.1 [15] Let $A = \{H_1, \ldots, H_n\}$ be a complex r-arrangement and let $\mathrm{A}^*_Z(A)$ be its Orlik-Solomon algebra over the integer ring Z. The map $a_H \mapsto (1/2\pi\sqrt{-1})\omega_H$ induces an isomorphism $\omega^* : \mathrm{A}^*_Z(A) \to H^*(M(A), Z)$ of graded Z-algebras, where $\omega_H = d\alpha_H / \alpha_H$ is the deferential 1-form for $H \in A$ and $H = \ker(\alpha)$.

Theorem 2.2 [19] *(Universal Coefficients Theorem for Cohomology)* For any commutative ring K and for $k \ge 0$, $H^k(M(A), K) \cong H^k(M(A), Z) \otimes Tor(H^{k+1}(M(A), Z), K)$, where $Tor(H^{k+1}(M(A), Z), K) = \ker(i^{k+1}, 1_K)$ and the short exact sequence;

$$0 \to R^{k+1} \xrightarrow{i^{k+1}} F^{k+1} \to H^{k+1}(M(A), Z) \to 0;$$

forms a free presentation of $H^{k+1}(M(A), Z)$ as generators F^{k+1} and relations R^{k+1}.

Definition 2.1 [15] We call the r-arrangements A_1 and A_2, A-equivalent if they have isomorphic Orlik-Solomon algebras.

Corollary 2.1 Any L_2-equivalent supersolvable r-arrangements are A-equivalent and have isomorphic cohomological groups.
Proof: It is a dirct result to theorems (1.2) and (2.1). □
Definition 2.2 Let A be a hypersolvable r-arrangement with Hp $\Pi = (\Pi_1, \ldots, \Pi_\ell)$. Define $\tau^* : \mathrm{A}^*_K(A) \to (\Pi)^*_K$ as follows; for $\mathrm{S} \in NBC(A)$, let $\tau^*(a_{\mathrm{S}}) = q_{\mathrm{S}}$ and from the universal property of the free module $\mathrm{A}^*_K(A)$, let τ^* be the unique homogeneous K-linear map of degree 0 which extend this assignment. Definitely τ^* is a monomorphism since $NBC(\mathrm{A}) \subseteq \mathrm{S}$.

Theorem 2.3 [12] and [13] Jambu's-Papadima's deformation theorem Let A be a hypersolvable r-arrangement such that $r < \ell$, i.e. A is not supersolvable. Then there is a vertical deformation $\{\widetilde{A}_t\}_{t \in C}$ of A in $C^r \times C^s = C^\ell$, where $s = \ell - r$ is the number of the sigular blocks of Π, such that for each $t \in C$, $\widetilde{A}_t$ is supersolvable, A and $\widetilde{A}_t$ are L_2-equivalent and they have isomorphic fundamental groups.

Theorem 2.4 Let A be a hypersolvable r-arrangement with Hp $\Pi = (\Pi_1, \ldots, \Pi_\ell)$. Then:

1. The partition complex $((\Pi)^*_K, \partial^*_\Pi)$ is acyclic,

where $\partial^*_\Pi : (\Pi)^*_K \to (\Pi)^*_K$ is a differentiation defined on $((\Pi)^*_K$ as $\partial^0_\Pi(q_{\{\}}) = 0$, $\partial^1_\Pi(q_H) = 1$ and for $2 \le k \le \ell$, $\partial^k_\Pi(q_\mathrm{S}) = \sum_{j=1}^{k}(-1)^{j-1} q_{\mathrm{S}_j}$, where S_j denote the subarrangement of $\mathrm{S} \in \mathrm{S}_k$ with the j^{th}-hyperplane of S deleted.

2. A is supersolvable if, and only if, $\mathrm{A}^*_K(A) \cong (\Pi)^*_K$.
3. A is supersolvable if, and only if, $H^*(M(A), Z) \cong (\Pi)^*_Z$.
4. If $r < \ell$ and $\{\tilde{A}_t\}_{t\in C}$ be the Jambu's-Papadima's vertical deformation of A, then $\{\tilde{A}_t\}_{t\in C}$ is a family of A-equivalent supersolvable arrangements, i.e. for all $t_1, t_2 \in C\setminus\{0\}$, $\mathrm{A}^*_K(\tilde{A}_{t_1}) \cong \mathrm{A}^*_K(\tilde{A}_{t_2}) \cong (\Pi)^*_K$. Thus, $H^*(M(\tilde{A}_{t_1}), Z) \cong H^*(M(\tilde{A}_{t_2}), Z) \cong (\Pi)^*_Z$.

Proof:

For 1. From the construction of the Hp Π, the first block Π_1 is a singleton. Hence the complex $((\Pi)^*_K, \partial^*_\Pi)$ is acyclic by lemma (3.86) of [15].

For 2 and 3. It is an application of theorem (1.2) and theorem (3.87) of [15] .

For 3. It is a direct result of theorem (1.3), since $\tilde{A}_{t_1}$ and $\tilde{A}_{t_2}$ are L_2-equivalent. □

Theorem 2.5 Let A be a hypersolvable r-arrangement with Hp $\Pi = (\Pi_1, \ldots, \Pi_\ell)$ such that $r < \ell$. Then the cohomological Z-ring $H^*(M(A)) = H^*(M(A), Z)$ and the Orlik-Solomon algebra $\mathrm{A}^*_Z(A)$ can be embedded as free Z-submodules of $(\Pi)^*_Z$ as follows;

1. for $0 \le k \le p(A)$, $H^k(M(A), Z) \cong \mathrm{A}^k_Z(A) \cong (\Pi)^k_Z$, and
2. for $p(A)+1 \le k \le r$, $H^k(M(A), Z) \cong \mathrm{A}^k_Z(A) \cong (\Pi)'^k_Z$, where

$$(\Pi)'^k_Z = \bigoplus_{\mathrm{S}\in \mathbf{NBC}_k} K q_\mathrm{S} \subseteq (\Pi)^k_Z;$$

is the free Z-submodule of $(\Pi)^k_Z$ with basis $\{q_\mathrm{S} \mid \mathrm{S} \in NBC_k(A)\}$. The following commutative diagram (2.1) explains that, where $(\omega^{-1})^* : H^*(M(A), Z) \xrightarrow{\sim} \mathrm{A}^*_Z(A)$ be the inverse of the isomorphism $\omega^* : \mathrm{A}^*_Z(A) \to H^*(M(A), Z)$ of graded Z-algebras given in theorem (2.1).

$$\begin{tikzcd}
0 \arrow[r, "\partial^{r+1}_H"] & H^r(M(A)) \arrow[r, "\partial^r_H"] \arrow[d, "(\omega^{-1})^r"'] & \cdots \arrow[r, "\partial^{p(A)+2}_H"] & H^{p(A)+1}(M(A)) \arrow[r, "\partial^{p(A)+1}_H"] \arrow[d, "(\omega^{-1})^{p(A)+1}"'] & H^{p(A)}(M(A)) \arrow[r, "\partial^{p(A)}_H"] \arrow[d, "(\omega^{-1})^{p(A)}"'] & \cdots \arrow[r, "\partial^1_H"] & H^0(M(A)) \arrow[r, "\partial^0_H"] \arrow[d, "(\omega^{-1})^0"'] & 0 \\
0 \arrow[r, "\partial^{r+1}_\mathrm{A}"] & \mathrm{A}^r_Z(A) \arrow[r, "\partial^r_\mathrm{A}"] \arrow[d, "\tau'^r"'] & \cdots \arrow[r, "\partial^{p(A)+2}_\mathrm{A}"] & \mathrm{A}^{p(A)+1}_Z(A) \arrow[r, "\partial^{p(A)+1}_\mathrm{A}"] \arrow[d, "\tau'^{p(\mathrm{A})+1}"'] & \mathrm{A}^{p(A)}_Z(A) \arrow[r, "\partial^{p(A)}_\mathrm{A}"] \arrow[d, "\tau^{p(\mathrm{A})}"'] & \cdots \arrow[r, "\partial^1_\mathrm{A}"] & \mathrm{A}^0_Z(A) \arrow[r, "\partial^0_\mathrm{A}"] \arrow[d, "\tau^0"'] & 0 \\
0 \arrow[r, "\partial^{r+1}_{\Pi'}"] & (\Pi)^r_Z \arrow[r, "\partial^r_{\Pi'}"] & \cdots \arrow[r, "\partial^{p(A)+2}_{\Pi'}"] & (\Pi)^{p(A)+1}_Z \arrow[r, "\partial^{p(A)+1}_{\Pi'}"] & (\Pi)^{p(A)}_Z \arrow[r, "\partial^{p(A)}_\Pi"] & \cdots \arrow[r, "\partial^1_\Pi"] & (\Pi)^0_Z \arrow[r, "\partial^0_\Pi"] & 0
\end{tikzcd};$$

Diagram (2.1)

Proof: It is a dirct result to our work in [2] and theorem (2.1). □

Corollary 2.2 Suppose we have the conclusions of theorem (2.5). Then Jambu's-Papadima's deformation, deformed the cohomological Z-ring $H^*(M(A)) = H^*(M(A), Z)$ and the Orlik-Solomon algebra $\mathrm{A}^*_Z(A)$ as follows;

1. for $0 \le k \le p(A)$, $H^k(M(A),Z) \cong \mathrm{A}_Z^k(A)$ is invariant under the deformation;
2. for $p(A)+1 \le k \le r$, $H^k(M(\tilde{A}_t),Z) \cong \mathrm{A}_Z^k(A)$ deformed into $H^k(M(\tilde{A}_t),Z) \cong \mathrm{A}_Z^k(\tilde{A}_t) \cong (\Pi)_Z^k$;
3. for $r+1 \le k \le \ell$, Jambu's-Papadima's deformation added exactly;

$$\sum_{i_1=1}^{\ell-k+1} \cdots \sum_{i_k=i_{k-1}+1}^{\ell} d_{i_1} \ldots d_{i_k};$$

k-NBC bases to obtain $H^k(M(\tilde{A}_t),Z) \cong \mathrm{A}_Z^k(\tilde{A}_t) \cong (\Pi)_Z^k$.

Proof: It is a direct result of the theorems (1.4), (2.1), (2.4) and (2.5). □

3 *Minimal cell decomposition of a supersolvable arrangement*

Definition 3.1 [16] Let X be a topological space with the following properties:

1. X is homotopy equivalent to a connected, finite type CW complex;
2. The homology groups $H_*(X)$ are torsion free;
3. The cup product $\cup : \bigwedge^* H^1(X) \to H^*(X)$ is surjective, and;

let p be a non negative integer. The space X is said to be p-minimal if X has the homotopy type of a CW complex $\mathbf{K}$ such that the number of k-cells in $\mathbf{K}$ is $b_k(X) = rk(H^k(X))$, for all $k \le p$. We call X minimal if it is p-minimal, for all p.

Definition 3.2 [6] Assume each of $G_1,\ldots,G_\ell$ be a group, and for $1 \le i < j \le \ell$, $\alpha_j^i : G_i \to Aut(G_j)$ satisfying the compatibility conditions, $\alpha_k^j(g_j^{\alpha_k^i(g_i)}) = (\alpha_k^i(g_i))^{-1}\alpha_k^j(g_j)\alpha_k^i(g_i)$, for $i < j < k$. Then, we define the iterated semi direct product of $G_1,\ldots,G_\ell$ with respect to the actions α_j^i to be the group $G = G_\ell \propto_{\alpha_\ell} G_{\ell-1} \propto_{\alpha_{\ell-1}} \cdots \propto_{\alpha_3} G_2 \propto_{\alpha_2} G_1$, where for each $1 \le k \le \ell$, the partial iteration $G^k = G_k \propto_{\alpha_k} G^{k-1}$ is defined by the homomorphism $\alpha_k : G^{k-1} \to Aut(G_k)$ with a restriction to G_k, $(\alpha_k)_{G_p} = \alpha_k^p : G_p \to Aut(G_k)$, $1 \le p \le k \le \ell$.

Definition 3.3 [15] and [22] Let A be a complex central essential r-arrangement with complement $M(A) \subseteq C^r$. Define a stratification $\mathfrak{I}$ of C^r (see [4]), as follows: For each $X \in L(A)$;

1. determine the arrangement $A^X = \{H \cap X; H \in A \setminus A_X \text{ and } H \cap X \ne \phi\} \subseteq X$, where $A_X = \{H \in A;\ X \subseteq H\} \subseteq A$, and
2. dfine M^X to be the complement of A^X in X.

Notice that the family $\{M^X\}_{X \in L(A)}$ forms a stratification of C^r with top dimensional stratum $M(A)$ and each strata is a convex relatively open sets of X. On the other hand, deduce that $cl(M^X) = X = M^X \cup \partial(M^X)$ splits into a disjoint union, where the boundary set of M^X is $\partial(M^X) = \bigcup_{H \in A \setminus A_X} (H \cap X)$.

Construction 3.1 Let A be a supersolvable ℓ-arrangement. Without loss of generality, assume that A is essential (see [12]). Then A has a maximal chain of modular elements say $C^\ell = X_0 < \cdots < X_\ell = \{0^\ell\}$, which induces a supersolvable composition series $\{H\} = A_{X_1} \subset \cdots \subset A_{X_\ell} = A$. We derived our supersolvable nice (completely factored) partition $\Pi = (\Pi_1, ..., \Pi_\ell)$, as follows;

1. $\Pi_1 = A_{X_1}$; and
2. for $2 \le k \le \ell$, $\Pi_k = A_{X_k} \setminus A_{X_{k-1}}$.

On the other hand, A is fiber type arrangement and the supersolvable composition series above induces a tower of fibrations;

$$M(A) = M(A_{X_\ell}) \xrightarrow{p_{\ell-1}} M(A_{X_{\ell-1}}) \xrightarrow{p_{\ell-2}} \cdots \xrightarrow{p_2} M(A_{X_2}) \xrightarrow{p_1} M(A_{X_1}) = M(H) = C_{/\{0\}};$$

with fiber F^k of p_k homeomorphic to C with d_k points removed and the fundamental group of the complement $\pi = \pi_1(M(A))$ admits a fashion of iterated semi direct product of finitely generated groups $\pi = F_{d_\ell} \propto_{\alpha_\ell} F_{d_{\ell-1}} \propto_{\alpha_{\ell-1}} \cdots \propto_{\alpha_3} F_{d_2} \propto_{\alpha_2} F_{d_1}$, where $F_{d_k} = \langle g_{1,k}, \ldots, g_{d_k,k} \rangle$ is free on d_q generators, i.e. for $1 \le i \le d_k$, the generator $g_{i,k}$ related to a k th hyperplane of the block Π_k via a topological order induced from the following structure of the fundamental group. It follows readily that the group π has a presentation;

$$\pi = \left\langle g_{i,k}, \begin{matrix} 1 \le i \le d_k \\ 1 \le k \le \ell \end{matrix} \middle| \alpha_k^{j,p}(g_{i,k}) = g_{j,p}^{-1} g_{i,k} g_{j,p}, p < k \right\rangle \ \ldots (3.1)$$

where each $\alpha_k^{j,p} = \alpha_k(g_{j,p}) \in Aut(F_{d_k})$.

Now, reordered the hyperplanes of A by an order induced the actions given by (3.1). It is clear that the ordering which induced from the structure of the fundamental group is compatible with our difinition of hypersolvable ordering and the fact that every 2-section of Π is a 2-NBC base of A. Express the structure of the supersolvable partition above in the structure of the cohomological groups as, $H^*(M(A), Z) \cong (\Pi)_Z^*$. We will construct a (finite type) minimal $K(\pi,1)$ CW-complex structure of $M(A)$ that given in ([21], 6.44 p. 95) induced from the presentation (3.1) above:

1. Partitioned C^ℓ by the stratification defined in definition (3.3);
2. Choose any point in $M(A)$, say e^0 and put $M(A)^0 = \{e^0\}$ to be the 0th-skeleton of $M(A)$.
3. Take $M(A)^1 = \vee_{k=1}^{\ell}(\vee_{i=1}^{d_k} S^1_{g_{i,k}}) = \vee_{H \in A} S^1$ to be the 1st- skeleton of $M(A)$. Geometricly, for each $H \in A$, the stratum M^H contains no point from any other hyperplane $H' \in A$, i.e. $M^H \cap H' = \phi$. On the other side of $M(A)$, we have a 1-cell e^1_H and an attaching mapping $\varphi^1_H : \partial e^1_H \to \{e^0\}$, attached the boundaries of e^1_H with e^0, i.e. we start in e^0, go around M^H and return into e^0 by e^1_H. Thus; $\pi_0(M(A), e^0) \cong \pi_0(M(A)^1, e^0) \cong 0$, since $M(A)$ is path connected space.
4. For the relations r_γ of the presentation (3.1);

$$0 \to \langle r_\gamma : \ 1 \le \gamma \le b_2(M(A)) \rangle \xrightarrow{\beta} \left\langle g_{i,k} : \ \begin{matrix} 1 \le i \le d_k \\ 1 \le k \le \ell \end{matrix} \right\rangle \xrightarrow{\alpha} \pi_1(M(A), e^0) \to 0;$$

we will choose maps $\varphi^2_\gamma : (S^1, s_0) \to (M(A)^1, e^0)$ representing $\beta(r_\gamma)$ and attach 2-cells e^2_r

by the maps φ_γ^2. Then;

$$M(A)^2 = M(A)^1 \coprod_{\substack{\gamma=1\\ \varphi_\gamma^2}}^{b_2(M(A))} e_\gamma^2 = M(A)^1 \coprod_{\substack{\gamma=1\\ \varphi_\gamma^2}}^{b_2(M(A))} S_\gamma^2 \text{ and;}$$

$\pi_1(M(A)^2, e^0) \cong \pi_1(M(A), e^0)$. Thus, $M(A)^2 / M(A)^1 = \vee_{\gamma=1}^{b_2(M(A))} S_\gamma^2$, see [21].

5. We will attach higher cells to kill off the higher homotopy group by induction as follows:

For $2 \le k \le \ell - 1$ Choose a set of generatores $\{g_\alpha^k\}_{\alpha=1}^{b_{k+1}(M(A))}$ for $\pi_k(M(A)^k, e^0)$ and maps $\varphi_\alpha^{k+1} : (S^k, s_0) \to (M(A)^k, e^0)$ representing g_α^k. Attach $(k+1)$-cells e_α^{k+1} by means of φ_α^k, $1 \le \alpha \le b_{k+1}(M(A))$, and let ;

$$M(A)^{k+1} = M(A)^k \coprod_{\substack{\alpha=1\\ \varphi_\gamma^k}}^{b_{k+1}(M(A))} e_\gamma^{k+1} = M(A)^k \coprod_{\substack{\alpha=1\\ \varphi_\gamma^k}}^{b_{k+1}(M(A))} S_\gamma^{k+1}.$$

Let $f_\alpha^{k+1} : (D^{k+1}, S^k, s_0) \to (M(A)^k, \varphi_\alpha^{k+1}(S^k), e^0)$ be the characteristic map of e_α^{k+1}. Then in the exact homotopy sequence;

$$\cdots \to \pi_{k+1}(M(A)^{k+1}, M(A)^k, e^0) \xrightarrow{d} \pi_k(M(A)^k, e^0) \to \pi_k(M(A)^{k+1}, e^0) \to 0;$$

we have d is an epimorphism. Therefore, $\pi_k(M(A)^{k+1}, e^0) \cong 0$ and;

$$\pi_r(M(A)^{k+1}, e^0) \cong \pi_r(M(A)^k, e^0), \text{ for } 0 \le r < k.$$

We take $M(A) = \bigcup_{k=0}^{\ell} X^k$ with the weak topology and

$$\pi_r(M(A), e^0) = \pi_r(M(A)^{r+1}, e^0) = \begin{cases} \pi_1(M(A)^2, e^0) & r = 1 \\ 0 & r \ne 1 \end{cases}.$$

We will introduce a topological interpretation to our question; "Why the structure of the completely factored partition Π that preserve the fundamental group structure, induced the number of the attached cells in each skeleton?" as follows:

For $2 \le k \le \ell - 1$, the inclusion map $i : M(A)^k \to M(A)$ of the k^{th}-skeleton induces an epimorphism $i_* : \pi_k(M(A)^k, e^0) \to \pi_k(M(A), e^0)$, i.e. $\pi_k(M(A)^k, e^0)$ and $\pi_k(M(A), e^0)$ have the same generators $\{g_\alpha^k\}_{\alpha=1}^{b_{k+1}(M(A))}$. We have the following exact sequnce;

$$\pi_{k+1}(M(A)^{k+1}, M(A)^k, e^o) \xrightarrow{d} \pi_k(M(A)^k, e^o) \xrightarrow{i_*} \pi_k(M(A)^{k+1}, e^o) = \pi_k(M(A), e^o) = 0.$$

Thus, d is an epimorphism, since $\operatorname{Im}(d) = \ker(i_*) = \pi_k(M(A)^k, e^o)$. It is known that $\pi_{k+1}(M(A)^{k+1}, M(A)^k, e^0)$ is free abelian group generated by the elements $\gamma.[f_\alpha^{k+1}]$, where $f_\alpha{}^k : (D^{k+1}, S^k, s_0) \to (M(A)^k, \varphi_\alpha^{k+1}(S^k), e^0)$ be the characteristic map of the $(k+1)$-cell e_α^{k+1}, for $1 \le \alpha \le b_{k+1}(M(A))$ and $\gamma \in \pi_1(M(A)^k, e^0) \cong \pi_1(M(A), e^0)$ and that caused by the actions of the first fundamental groups on the higher homotopy groups which has a fashion as $Z\pi$-module, where $Z\pi$ is the group ring of $\pi = \pi_1(M(A), e^0)$. On the other hand, $\ker(i_*) = \operatorname{Im}(d)$ is $\pi_k(M(A)^k, e^0)$ generated by the elements $\gamma.[\theta_\alpha^{k+1}]$, where $\theta_\alpha{}^{k+1} : (S^k, s_0) \to (\varphi_\alpha^{k+1}(S^k), e^0)$ be the attaching map of the $(k+1)$-cell $e_\alpha{}^{k+1}$ to kill off a k-hole by means of φ_α^{k+1}, for $1 \le \alpha \le b_{k+1}(M(A)) = \sum_{i_1=1}^{\ell-k} \cdots \sum_{i_k = i_{k-1}+1}^{\ell} d_{i_1} \ldots d_{i_{k+1}}$ and $\gamma \in \pi_1(M(A)^k, e^0) \cong \pi_1(M(A), e^0)$. Therefore; the k^{th}-higher homotopy group has the following presentation:

$$\pi_k(M(A),e^0) = \pi_k(M(A)^{k+1},e^o) = \pi_k(M(A)^k,e^0)\setminus\ker(i^*)$$
$$= \left\langle g_\alpha^k \quad ;1 \le \alpha \le b_{k+1}(M(A)) \quad | \quad \gamma.[\theta_\alpha^{k+1}] \quad \begin{matrix} 1 \le \alpha \le b_{k+1}(M(A)) \\ \gamma \in \pi_1(M(A),e^0) \end{matrix} \right\rangle .$$

Recall theorem (4.4), we mirror the properties of Π in the structure of the cohomological group of the complement as a partition Z-module given in section four as;

$$H^*(M(A),Z) = \overset{\ell}{\underset{k=1}{\otimes}} H^*(F^k,Z) \cong (\Pi)_Z^* = \overset{\ell}{\underset{k=1}{\otimes}} (\Pi_k)_Z^* = \overset{\ell}{\underset{k=1}{\oplus}} (\Pi)_Z^k \, ;$$

where the cup product $\bigcup : \wedge^* H^1(M(A),Z) \to H^*(M(A),Z)$ is surjective. For $2 \le k \le \ell - 1$, the $(k+1)$-cells e_α^{k+1} is related to a generator of $H^{k+1}(M(A),Z)$, which is by above assertion, is related to a $(k+1)$-section of Π and it is a NBC base of A via the fundemantal group structure. Thus, for $X \in L(A)$, the subarrangement A_X of A has NBC bases via the fundemantal group ordering. Of the complement, the stratum M^X forms our holes and the number of the generaters and related relations there to kill off the higher holes with keep in mind one dimensional holes, is equal to the number of sections of Π there.

Remark 3.1 We emphasize that Switzer in [21] showed that, for any topological space X, one can construct a CW complex X' (as showed in construction (3.1)) and a weak homotopy equivalence $f : X \to X'$ and this construction is unique up to homotopy and Whitehead proved that, if $f : X \to X'$ is a weak homotopy equivalence between CW complexes, then X and X' of the same homotopy type.

Example 3.1 Let A be a central ℓ-arrangement of hyperplanes of C^ℓ. Randell in [18], choosed a 3-dimensional linear subspace U of C^ℓ such that it is in general position with respect to the variety $(\bigcup_{H \in A} H)$ in the sense that if, $\dim_C(H_{i_1} \cap \cdots \cap H_{i_r}) = \ell - p$, then $\dim_C(U \cap H_{i_1} \cap \cdots \cap H_{i_r}) = 3 - p$. Let $N(A)$ be the quotient space of $M(A)$ by the Hopf action of $C^* = C \setminus \{0\}$. We will obtain a curve $D \cong U \cap (\bigcup_{H \in A} H)/C^* \subseteq CP^2$. By applying Zariski theorem, Randell showed that $\pi_1(N(A)) = \pi_1(CP^2 - D)$. The curve D consists of $|A|$-projective lines $\{D_H\}_{H \in A}$, not necessarily in general position. For each D_H, we introduce a generator g_H, $H \in A$. For each point p_j, $1 \le j \le k$, of intersection of $D_{i_1}, \ldots, D_{i_t}$ lines, we introduce the torus link relations of type *(t,t)*;

$$R_X = \{g_{i_1} \ldots g_{i_t} = g_{i_2} \ldots g_{i_t} g_{i_1} = \cdots = g_{i_t} g_{i_1} \ldots g_{i_{t-1}}\} .$$

Then $\pi_1(N(A)) = \langle g_H ; H \in A | \quad g_1 \ldots g_n,\ R_X,\ X \in L_2(A) \rangle$, with respect an order defined on the hyperplanes induced from the Randell construction. Then, he gives the fundamental group of complement by the following exact sequence;

$$0 \to \pi_1(C^*) = Z \to \pi_1(M(A)) \to \pi_1(N(A)) \to 0 .$$

Suppose A be a complex central essential 3-arrangement. Then the decone dA over A is an arrangement in CP^2 and each hyperplane $H \in A$, forms a line D_H of CP^2. On the other hand, each $X \in L_2(A)$ deconed into a point $dX = p_X$ of CP^2 and it represents an intersection point of $D_{H_{i_1}}, \ldots, D_{H_{i_t}}$, where $A_X = \{H_{i_1}, \ldots, H_{i_t}\}$. Recall the stratification given in def. (3.3). Each 2-relation $\alpha_k^{j,p}(g_{i,k}) = g_{j,p}^{-1} g_{i,k} g_{j,p}$ given in presentation (3.1), related to a 2-section $\{H_{j,p}, H_{i,k}\}$ for some $1 \le p < k \le \ell$, $1 \le j \le d_p$ and $1 \le i \le d_k$. We are now in location $M^{\{H_{j,p}, H_{i,k}\}}$ and there are two possible type of a flat $X \in L_2(A)$ with $H_{i,k}, H_{j,p} \in A_X$ induced from Π. So, to compute the actions we will used Randell presentation [18], as follows:

1. If $X = \{H_{j,p}, H_{i,k}\} \in L(A)$. Then, $M^{\{H_{j,p},H_{i,k}\}} \bigcap_{H \notin A_X} H = \phi$ and the action $\alpha_k^{j,p}(g_{i,k}) = g_{i,k}$ is trivial and the relation will be a usual commutator relation, i.e. $g_{j,p}^{-1} g_{i,k} g_{j,p} = g_{i,k} \Rightarrow g_{i,k}^{-1} g_{j,p}^{-1} g_{i,k} g_{j,p} = 0$, i.e. we have a torus relation as the following figure:

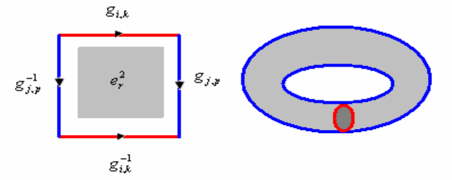

Figure (3.1)

For example, if $Q(A) = xyz$, be the defining polynomial of the Boolean arrangement A, then A is supersolvable with supersolvable partition has exponent vector $d = (1,1,1)$ and the fundamental group of A is Z^3;

$$\pi_1(M(A), e^0) = \left\langle g_1, g_2, g_3 \;\middle|\; \begin{matrix} g_2 = g_1^{-1} g_2 g_1 \\ g_3 = g_1^{-1} g_3 g_1 \\ g_3 = g_2^{-1} g_3 g_2 \end{matrix} \right\rangle.$$

Then, it has second skeleton as;

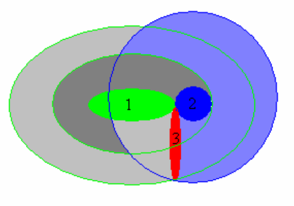

Figure (3.1)

Then we will attach one three dimensional cell to kill off the second higher homotopy group as showed in construction (3.1).

2. If $X = \{H_{j,p}, H_{l_1,k}, \ldots, H_{l_t,k}\} \in L(A)$, $1 \le l_2 < \cdots < l_t \le d_k$ via the fundamental group ordering such that $H_{i,k} = H_{l_s,k}$ for some $2 \le s \le t$ and we have the following relation:

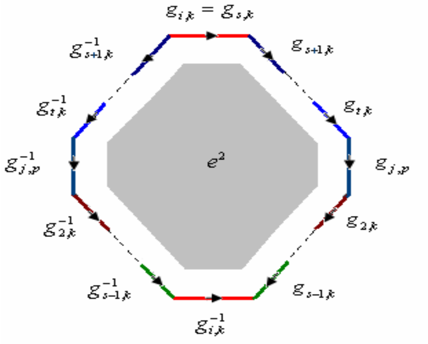

Figure (3.2)

For example, if $t = 3$, we have the following relations and attaching mapping via those relations;

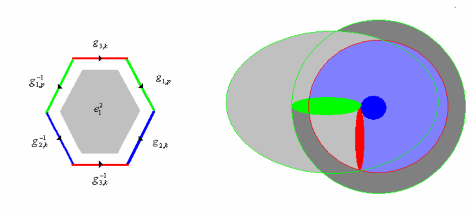

Figure (3.3.a)

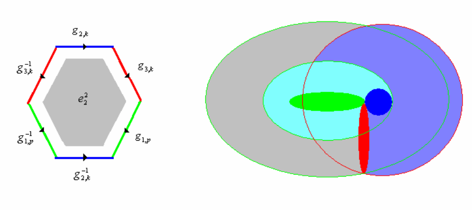

Figure (3.3.b)

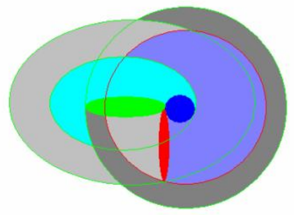

Figure (3.3.c)

The CW complex that given in fig. (3.3.c) is agreement with the minimal CW complex for the complexified arrangement A of C^2 with defining polynomial $Q(A) = xy(x-y)$, which is of the same homotopy type of $(S^1 \vee S^1) \times S^1$.

That is, we can used Randell presentation to construct the second skeleton, then we will attach higher cells to kill off the higher homotopy group as showed in construction (3.1).

4 A topological interpretation for vanishing of higher homotopy groups of a hypersolvable arrangements

In the remainder of this section we assume A to be a hypersolvable r-arrangement with Hp $\Pi = (\Pi_1, ..., \Pi_\ell)$ and exponent vector is $d = (d_1, ..., d_\ell)$, such that $r < \ell$, i.e. A is not supersolvable. For each $H \in A$, Jambu and Papadima identified the defining plynomial $\alpha_H : C^r \to C$, $\alpha_H(x_1, \dots, x_r) = \alpha_1^H x_1 + \cdots + \alpha_r^H x_r$, as a point $\alpha_H = (\alpha_1^H, \dots, \alpha_r^H) \in CP^{*r-1}$ in the projective dual space CP^{*r-1} and they embeded it in $CP^{*\ell-1}$ by the trivial lift $\hat{\alpha}_H = (\alpha_1^H, \dots, \alpha_r^H, 0, \dots, 0) \in CP^{*\ell-1}$. Thus they defined an ℓ-arrangement $\hat{A}$, such that $\hat{H} \in \hat{A}$ has defining polynomial $\hat{\alpha}_H : C^\ell \to C$ as, $\hat{\alpha}_H(x_1, \dots, x_\ell) = \alpha_1^H x_1 + \cdots + \alpha_r^H x_r + 0x_{r+1} + \cdots + 0x_\ell$. Let $\alpha_{\tilde{H}_t} = (\alpha_1^H, \dots, \alpha_r^H, \lambda_1^{\tilde{H}_t}, \dots, \lambda_s^{\tilde{H}_t}) \in CP^{*\ell-1}$ represents the point in $CP^{*\ell-1}$ related to the deformed hyperplane $\tilde{H} \in \tilde{A}_t$ by Jambu's Papadima's deformation method. It is known, $M(A)$ is a finite type CW complex. For a given cell complex K on $M(A)$, the space;

$$M(A) \times \{\underbrace{(0, \dots, 0)}_{s\text{ times}}\} = M(A) \times 0^s;$$

include a structuer of finite type CW complex induced from $K \times \{e^0\}$.

Proposition 4.1 [1]

i. $H \cong H \times 0^s \subseteq \tilde{H}$ is a strong deformation retract of $\hat{H}$.

ii. $\bigcup_{H \in A} H \cong (\bigcup_{H \in A} H) \times 0^s \subseteq \bigcup_{H \in A} \tilde{H}_t$ is a strong deformation retract of $\bigcup_{H \in A} \hat{H}$.

iii. $\bigcap_{H \in A_X} H \cong (\bigcap_{H \in A_X} H) \times 0^s \subseteq \bigcap_{H \in A_X} \tilde{H}_t$ is a strong deformation retract of $\bigcap_{H \in A_X} \hat{H}$, for all $X \in L(A)$. That is we can embed the lattice intersection $L(A)$ in $L(\tilde{A}_t)$.

iv. $M(A) \cong M(A) \times 0^s \subseteq M(\tilde{A}_t)$ is a strong deformation retract of $M(\hat{A})$.

Remark 4.1 Thus, $(M(\tilde{A}_t), M(A))$ is a topological pair. Hence from ([21], p. 38) the relative homotopy groups $\pi_n(M(\tilde{A}_t), M(A), \underline{x})$, $n \geq 0$ and ($\underline{x} \in M(A) \times 0^s$) fit into the following long exact sequence:

$$\to \pi_n(M(A), \underline{x}) \xrightarrow{i_*} \pi_n(M(\tilde{A}_t), \underline{x}) \xrightarrow{j_*} \pi_n(M(\tilde{A}_t), M(A), \underline{x}) \xrightarrow{\partial} \pi_{n-1}(M(A), \underline{x}) \to \cdots \xrightarrow{\partial} \pi_0(M(\tilde{A}_t), \underline{x});$$

which is called the exact homotopy sequence of the pair $(M(\tilde{A}_t), M(A))$. Since $M(\tilde{A}_t)$ is path connected, we have $\pi_0(M(\tilde{A}_t)) = 0$ is the trivial group, $\pi_n(M(\tilde{A}_t)) = 0$ for all $n \geq 2$, $\pi_1(M(\tilde{A}_t)) = F_{d_\ell} \propto_{\alpha_\ell} F_{d_{\ell-1}} \propto_{\alpha_{\ell-1}} \cdots \propto_{\alpha_3} F_{d_2} \propto_{\alpha_2} F_{d_1}$ and $\pi_1(M(\tilde{A}_t)) \cong \pi_1(M(A))$. Then the long exact sequence above becomes:

$$\to \pi_n(M(A), \underline{x}) \xrightarrow{i_*} 0 \xrightarrow{j_*} \pi_n(M(\tilde{A}_t), M(A), \underline{x}) \xrightarrow{\partial} \pi_{n-1}(M(A), \underline{x}) \xrightarrow{i_*} 0 \to \cdots$$

$$\cdots \to 0 \xrightarrow{j_*} \pi_2(M(\tilde{A}_t), M(A), \underline{x}) \xrightarrow{\partial} \pi_1(M(A), \underline{x}) \xrightarrow{\sim} \pi_1(M(\tilde{A}_t), \underline{x}) \xrightarrow{j_*} \pi_1(M(\tilde{A}_t), M(A), \underline{x}) \xrightarrow{\partial} 0$$

Proposition 4.2 [1] The topological pair $(M(\tilde{A}_t), M(A))$ is 2-connected and for all $n > 2$, $\pi_n(M(\tilde{A}_t), M(A)) \cong \pi_{n-1}(M(A))$. Therefore, $(M(\tilde{A}_t), M(A))$ is n-connected if, and only if, $\pi_m(M(A))$ vanishes for all $2 \leq m \leq n$.

Theorem 4.1 [1] Let A be a hypersolvable r-arrangement with Hp $\Pi = (\Pi_1, ..., \Pi_\ell)$ such that $r < \ell$ and let $\{\tilde{A}_t\}_{t \in C}$ be the Jambu's-Papadima's vertical deformation of A. Then:

1. For $1 \leq k \leq r$, the number of k-cells is;

$$\left|M^k(A)\right| = b_k(M(A)) = rk(H^k(M(A), Z)) \leq \sum_{i_1=1}^{\ell-k+1} \cdots \sum_{i_k=i_{k-1}+1}^{\ell} d_{i_1} \ldots d_{i_k}.$$

2. For $2 \leq k \leq p(A) < r$, the number of k-cells is;

$$\left|M^k(A)\right| = b_k(M(A)) = rk(H^k(M(A), Z)) = \sum_{i_1=1}^{\ell-k+1} \cdots \sum_{i_k=i_{k-1}+1}^{\ell} d_{i_1} \ldots d_{i_k}.$$

3. the number of $p(A)+1$-cells is;

$$\left|M^{p(A)+1}(A)\right| = b_{p(A)+1}(M(A)) = rk(H^{p(A)+1}(M(A), Z)) = \sum_{i_1=1}^{\ell-p(A)} \cdots \sum_{i_k=i_{k-1}+1}^{\ell} d_{i_1} \ldots d_{i_{p(A)+1}} - \left|S_{p(A)+1}(A) \cap BC_{p(A)+1}(A)\right|$$

4. For $p(A) + 2 \leq k \leq r$, the number of k-cells is;

$$\left|M^k(A)\right| = b_k(M(A)) = rk(H^k(M(A), Z)) = \sum_{i_1=1}^{\ell-k+1} \cdots \sum_{i_k=i_{k-1}+1}^{\ell} d_{i_1} \ldots d_{i_k} - \left|\overline{S_k(A)}\right|;$$

where $\overline{S_k(A)}$ forms the set of all k-sections that is not k-NBC bases.

That is the number of the ($p(A)+1$)- cells not enough to kill off all the;

$$\left(\sum_{i_1=1}^{\ell-p(A)} \cdots \sum_{i_k=i_{k-1}+1}^{\ell} d_{i_1} \ldots d_{i_{p(A)+1}}\right),\ p(A)\text{-holes.}$$

Theorem 4.2 $M(\tilde{A}) \approx M(A)^{p(A)} \coprod_{j=p(A)+1}^{\ell} \big(\coprod_{B \in S_j(A)} S_B^j \big)$ via attaching maps.

That is the deformation method deforms the complement $M(A)$ to $M(\tilde{A})$ by attaching;

$$\left| S_{p(A)+1}(A) \cap BC_{p(A)+1}(A) \right| \ (p(A)+1)\text{-cells and } \Big(\sum_{i_1=1}^{\ell-k+1} \cdots \sum_{i_k=i_{k-1}+1}^{\ell} d_{i_1} \ldots d_{i_k} \Big) \ k\text{-cells;}$$

for each $p(A)+2 \le k \le \ell$, these cells arises from destroying all sections of length greater than $p(A)$ of Π which is not NBC base of A, to kill off all the non-vanishing higher homotopy groups $\pi_k(M(A)), \forall k \ge p(A)$.

Proof: Papadima and Suciu proved that $M(A)$ and $M(\tilde{A}_t)$ have the same $p(A)^{\text{th}}$-skeletons and they have isomorphic k^{th}-higher homotopy groups $\pi_k(M(A)) \cong \pi_k(M(\tilde{A}_t)) \cong 0$, for $2 \le k < p(A) < r$. Thus $\pi_{p(A)}(M(\tilde{A}_t), M(A)) \cong \pi_{p(A)-1}(M(A)) \cong 0$.

Recall construction (3.1) as a (finite type) minimal CW complex of $M(\tilde{A}_t)$ and choose $e_0^A \in M(A)$ as the 0^{th}-cell of $M(\tilde{A}_t)$. Since $M(A)$ and $M(\tilde{A}_t)$ have the same $p(A)^{\text{th}}$-skeletons, let $i_k : M(A)^k \to M(\tilde{A}_t)^k$ and $r_k : M(\tilde{A}_t)^k \to M(A)^k$ be the cellular homotopy equivalences. Thus, the induced homomorphism $r_{k*} : \pi_k(M(\tilde{A}_t)^k, e_A^0) \to \pi_k(M(A)^k, e_A^0)$ form an isomorphism. First introduce the construction (3.1), as our construction of the k^{th}-skeleton of $M(A)$, $0 \le k \le p(A)$. Now we will attach higher cells to kill off the higher homotopy group by induction as follows:

For $k = p(A)$: Since $M(A)$ and $M(\tilde{A}_t)$ have the same $p(A)^{\text{th}}$-skeletons. Hence, the induced homomorphism $r_{p(A)*} : \pi_{p(A)}(M(\tilde{A}_t)^{p(A)}, e_A^0) \to \pi_{p(A)}(M(A)^{p(A)}, e_A^0)$ form an isomorphism. So, we can embed the set of generatores $\{g_\alpha^{p(A)}\}_{\alpha=1}^{b_{p(A)+1}(M(\tilde{A}_t))}$ for $\pi_{p(A)}(M(\tilde{A}_t)^{p(A)}, e_A^0)$ and the maps $\varphi_\alpha^{p(A)+1} : (S^{p(A)}, s_0) \to (M(\tilde{A}_t)^{p(A)}, e^0)$ representing $g_\alpha^{p(A)}$ of $\pi_{p(A)}(M(A)^{p(A)}, e_A^0)$ by meaning of $r_{p(A)*}(g_\alpha^{p(A)})$ and;

$$r_{p(A)*}\varphi_\alpha^{p(A)+1} : (S^{p(A)}, s_0) \to (M(A)^{p(A)}, e^0),\ 1 \le \alpha \le \sum_{i_1=1}^{\ell-p(A)} \cdots \sum_{i_k=i_{k-1}+1}^{\ell} d_{i_1} \ldots d_{i_{p(A)+1}}.$$

We have the following exact sequence;

$$\begin{tikzcd}
\pi_{p(A)+1}(M(A)^{p(A)+1}, M(A)^{p(A)}, e_A^0) \arrow[r, "d"] & \pi_{p(A)}(M(A)^{p(A)}, e_A^0) \arrow[r, "i'_*"] \arrow[d, shift right, "i_*"'] & \pi_{p(A)}(M(A), e_A^0) \neq 0 \\
& \pi_{p(A)}(M(\tilde{A}_t)^{p(A)}, e_A^0) \arrow[u, shift right, "i_*^{-1} = r_*"'] &
\end{tikzcd}$$

where the inclusion i'_* is an epimorphism, i.e. $\pi_{p(A)}(M(A)^{p(A)}, e_A^0)$ and $\pi_{p(A)}(M(A), e_A^0)$ have the same generators $\{r_{p(A)*}(g_\alpha^{p(A)})\}_{\alpha=1}^{b_{p(A)+1}(M(\tilde{A}_t))}$. And $\ker(i'_*) = \mathrm{Im}(d)$ is a subgroup of $\pi_{p(A)}(M(A)^{p(A)}, e_A^0)$ generated by the elements $\gamma.[\theta_\alpha^{p(A)+1}]$, where $\theta_\alpha^{p(A)+1} : (S^{p(A)}, s_0) \to (\varphi_\alpha^{p(A)+1}(S^{p(A)}), e^0)$ be the attaching map of the $(p(A)+1)$-cell $r(e_\alpha^{p(A)+1})$ to kill off a $p(A)$-hole by means of $r\varphi_\alpha^{p(A)+1}$, for $1 \le \alpha \le b_{p(A)+1}(M(A)) < \sum_{i_1=1}^{\ell-p(A)} \cdots \sum_{i_k=i_{k-1}+1}^{\ell} d_{i_1} \ldots d_{i_{p(A)+1}}$ and $\gamma \in \pi_1(M(A)^m, e^0) \cong \pi_1(M(A), e^0)$. The group $\pi_{p(A)+1}(M(A)^{p(A)+1}, M(A)^{p(A)}, e^0)$ is free abelian group generated by the elements $\gamma.[f_\alpha^{p(A)+1}]$, where;

$$f_\alpha^{p(A)+1} : (D^{p(A)+1}, S^{p(A)}, s_0) \to (M(A)^{p(A)}, \varphi_\alpha^{p(A)+1}(S^{p(A)}), e^0);$$

be the characteristic map of the ($p(A)+1$)-cell for $1 \le \alpha \le b_{p(A)+1}(M(A))$ and $\gamma \in \pi_1(M(A), e^0)$. It is clear that $\ker(i'_*) \ne \pi_{p(A)}(M(A)^{p(A)}, e_A^0)$, since $\pi_{p(A)}(M(A), e_A^0) \ne 0$. Now, assume that $r_{p(A)*}(g_\beta^{p(A)})$ be the generators of $\pi_{p(A)}(M(A)^{p(A)}, e_A^0)$ which is not a member of $\ker(i'_*)$, (i.e. it is not killed off), and the number of such generators is $q = \left| S_{p(A)+1}(A) \cap BC_{p(A)+1}(A) \right|$, the number of all ($p(A)+1$)-sections that is not NBC bases via the fundamental group order.

Let us retern into $\pi_{p(A)}(M(\tilde{A}_t)^{p(A)}, e_A^0)$ by the homotopy equivalence, $i_{p(A)*} : \pi_{p(A)}(M(A)^{p(A)}, e_A^0) \to \pi_{p(A)}(M(\tilde{A}_t)^{p(A)}, e_A^0)$, the set of generatores $\left\{ g_\beta^{p(A)} \right\}_{\beta=1}^{q}$ generates a subgroup of $\pi_{p(A)}(M(\tilde{A}_t)^{p(A)}, e_A^0)$. Recall the construction (3.1), via Jambu's and Papadima's deformation mathod, we will attach $(p(A)+1)$-cells $e_\beta^{p(A)+1}$ by means of $\varphi_\beta^{p(A)}$, $1 \le \beta \le q$, to kill off all the $(p(A)+1)$-holes to make the first non-vanishing higher homotopy group $\pi_{p(A)}(M(A)^{p(A)+1}, e_A^0) \cong \pi_{p(A)}(M(A), e_A^0) \cong \pi_{p(A)+1}(M(\tilde{A}), M(A), e_A^0)$ vanished, since $\pi_{p(A)}(M(\tilde{A}_t)^{p(A)+1}, e_A^0) \cong \pi_{p(A)}(M(\tilde{A}_t), e_A^0) \cong 0$. Thus;

$$\begin{aligned} M(\tilde{A}_t)^{p(A)+1} &= M(\tilde{A}_t)^{p(A)} \coprod\limits_{\substack{\alpha=1 \\ \varphi_\alpha^{p(A)}}}^{b_{p(A)+1}(M(\tilde{A}_t))} e_\alpha^{p(A)+1} \\ &\approx \left(M(A)^{p(A)} \coprod\limits_{\substack{\alpha=1 \\ \varphi_\alpha^{p(A)}}}^{b_{p(A)+1}(M(\tilde{A}_t))-q} r(e_\alpha^{p(A)+1}) \right) \coprod\limits_{\substack{\beta=1 \\ \varphi_\beta^{p(A)}}}^{q} r(e_\beta^{p(A)+1}) \\ &\approx M(A)^{p(A)+1} \coprod\limits_{\substack{\beta=1 \\ \varphi_\beta^{p(A)}}}^{q} r(e_\beta^{p(A)+1}) \\ &\approx M(A)^{p(A)+1} \coprod\limits_{\substack{\beta=1 \\ \varphi_\beta^{p(A)}}}^{q} S_\beta^{p(A)+1} \end{aligned}$$

and $\pi_r(M(\tilde{A}_t), e^0) = \pi_r(M(\tilde{A}_t)^{p(A)+1}, e^0) = \begin{cases} \pi_1(M(A), e^0) & r = 1 \\ 0 & 2 \le r \le p(A) \end{cases}$.

<u>For</u> <u>$p(A)+2 \le k \le \ell - 1$</u>**:** As in construction (3.1), we will kill off all the higher holes by the same mathod and we get;

$$M(\tilde{A}_t)^{k+1} = M(\tilde{A}_t)^{k} \coprod_{\substack{\alpha=1\\ \varphi_\alpha^k}}^{b_{k+1}(M(\tilde{A}_t))} e_\alpha^{k+1}$$

$$\approx \left(\left((M(A)^{p(A)} \coprod_{\substack{\alpha=1\\ \varphi_\alpha^k}}^{b_{p(A)+1}(M(\tilde{A}_t))-q} r(e_\alpha^{p(A)+1}) \right) \coprod_{\substack{\beta=1\\ \varphi_\beta^k}}^{q} r(e_\beta^{p(A)+1}) \right) \coprod_{j=p(A)+2}^{k} \left(\coprod_{\substack{\zeta=1\\ \varphi_\zeta^k}}^{b_{k+1}(M(\tilde{A}_t))} e_\zeta^{p(A)+1} \right)$$

$$\approx \left(M(A)^{p(A)+1} \coprod_{\substack{\beta=1\\ \varphi_\beta^k}}^{q} r(e_\beta^{p(A)+1}) \right) \coprod_{j=p(A)+2}^{k} \left(\coprod_{\substack{\zeta=1\\ \varphi_\zeta^k}}^{b_{k+1}(M(\tilde{A}_t))} e_\zeta^{p(A)+1} \right)$$

$$\approx \left(M(A)^{p(A)+1} \coprod_{\substack{\beta=1\\ \varphi_\beta^k}}^{q} S_\beta^{p(A)+1} \right) \coprod_{j=p(A)+2}^{k} \left(\coprod_{\substack{\zeta=1\\ \varphi_\zeta^k}}^{b_{k+1}(M(\tilde{A}_t))} S_\zeta^{p(A)+1} \right)$$

and $\pi_r(M(\tilde{A}_t), e^0) = \pi_r(M(\tilde{A}_t)^{k+1}, e^0) = \begin{cases} \pi_1(M(A), e^0) & r=1 \\ 0 & 2 \le r \le k \end{cases}$. It follows immediately that;

$$\pi_r(M(\tilde{A}), e^0) = \pi_r(M(\tilde{A})^{r+1}, e^0) = \begin{cases} \pi_1(M(A)^2, e^0) & r=1 \\ 0 & r \ne 1 \end{cases}. \quad \square$$

Example 4.1 Let A be a hypersolvable 3-arrangement with defining with defining polynomial;

$$Q(A) = (y-x+z)(y+x+z)z(y+3z)(y+2z)y(y-z);$$

Hp $\Pi = (\Pi_1, \Pi_2, \Pi_3, \Pi_4) = (\{H_1\}, \{H_2\}, \{H_3\}, \{H_4, H_5, H_6, H_7\})$ with the exponent vector $d = (1,1,1,4)$, and the Poincaré polynomial;

$$P(A,s) = 1 + 7s + 15s^2 + 9s^3 = (1+s)(1+3s)^2.$$

The supersolvable deformed 4-arrangement $\tilde{A}_t$ of A, has Poincaré polynomial;

$$P(\tilde{A}_t, s) = 1 + 7s + 15s^2 + 13s^3 + 4s^4.$$

Let dA be the deconing 2-arrangement in CP^2 with defining polynomial $Q(dA) = (y-x+1)(y+x+1)(y+3)(y+2)y(y-1)$.

Let us first Partitioned C^3 by the stratification defined in definition (3.3). To simplify notation we write *i* instead of H_i, for $1 \le i \le 7$, and we write *j* instead of X_j, for $8 \le j \le 19$, as shown in the following figure:

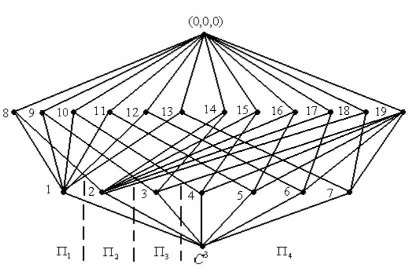

Figure (4.1)

Therefore, we will intersect each stratum M^X with the unit sphere S^5 in $C^3 \cong R^6$, to induced the stratum M^{dX} of CP^2 and the following configuration embedded the intersection lattice above (fig. (4.1)), in CP^2:

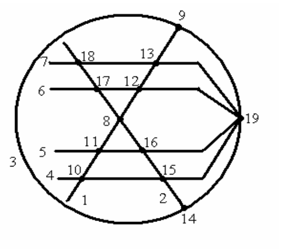

Figure (4.2)

From theorem (1.4), $p(A)=2$ and $\pi_2(M(A))$ will be the first non vanishing higher homotopy group. Therefore, we will construct the second skeleton $M(A)^2$ of $M(A)$ by applying construction (3.1), as shown in theorem (4.2):

1. Choose any point in $M(A)$, say e^0 and put $M(A)^0=\{e^0\}$ to be the 0th-skeleton of $M(A)$.
2. Take $M(A)^1=\vee_{k=1}^{7}S_k^1$ to be the 1st- skeleton of $M(A)$. Thus;

$$\pi_0(M(A),e^0)\cong\pi_0(M(A)^1,e^0)\cong 0;$$

   since $M(A)$ is path connected space.

3. For the relations r_γ of the presentation (3.1);

$$\pi_1(M(A),e^0)=\left\langle g_{i,k},\ \begin{matrix}1\le i\le d_k\\ 1\le k\le 4\end{matrix}\ \middle|\ \alpha_k^{j,p}(g_{i,k})=g_{j,p}^{-1}g_{i,k}g_{j,p},\ \begin{matrix}1\le j\le d_p\\ p<k\end{matrix}\right\rangle$$

$$=\langle g_1,\dots,g_7 \mid\ g_1\dots g_7,\ R_j,\ 8\le j\le 19\rangle\times Z$$

$$0\to\langle r_\gamma:\ 1\le\gamma\le 15\rangle\xrightarrow{\beta}\left\langle g_{i,k}:\ \begin{matrix}1\le i\le d_k\\ 1\le k\le 4\end{matrix}\right\rangle\xrightarrow{\alpha}\pi_1(M(A),e^0)\to 0;$$

   we will choose maps $\varphi_\gamma^2:(S^1,s_0)\to(M(A)^1,e^0)$ representing $\beta(r_\gamma)$ and attach 2-cells e_r^2 by the maps φ_γ^2. Then;

$$M(A)^2=M(A)^1\coprod_{\substack{\gamma=1\\ \varphi_\gamma^2}}^{15}e_\gamma^2=(\vee_{k=1}^{7}S_k^1)\coprod_{\substack{\gamma=1\\ \varphi_\gamma^2}}^{15}S_\gamma^2 \text{ and;}$$

   $\pi_1(M(A)^2,e^0)\cong\pi_1(M(A),e^0)$. Thus, $M(A)^2/M(A)^1=\vee_{\gamma=1}^{15}S_\gamma^2$. The important point to note here is for the ordering on the hyperplanes can be deduced from [3]. Then we have eleven 2-strata M^j associated to $8\le j\le 18$, with trivial action $g_{i,k}=g_p^{-1}g_{i,k}g_p$, for $1\le k\le 4$, $p=1,2$, $p<k$ and $1\le i\le d_k$, as follows:

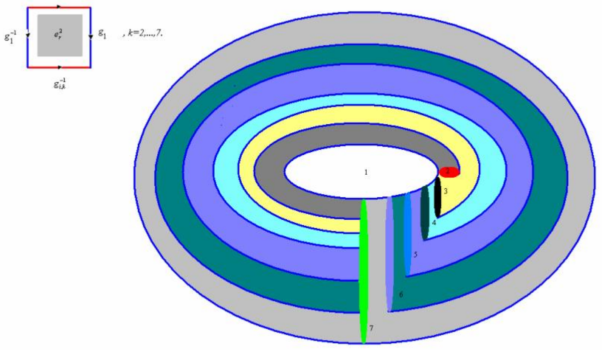

Figure (4.3)

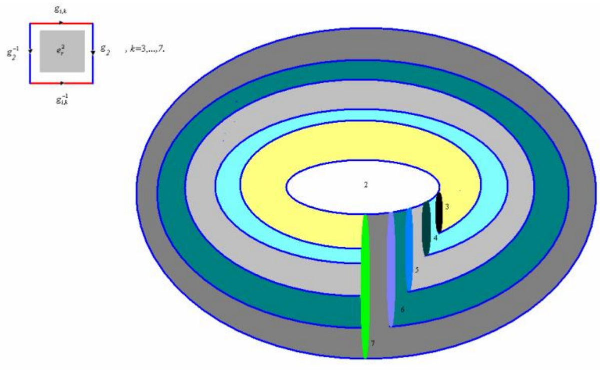

Figure (4.4)

Where, for $j = 19$, we have four relations as given in fig. (3.2), for $k = 4$, $p = 3$ and $4 \le i \le 7$.

We will attach higher cells of dimension 3 to kill off the first non vanishing higher homotopy group $\pi_2(M(A), e^0)$ via Jambu's and Papadima's deformation method as follows:

For each $t \in C^*$, $M(A)$ and $M(\tilde{A}_t)$ have the same second skeletons. Let $i_k : M(A)^k \to M(\tilde{A}_t)^k$ and $r_k : M(\tilde{A}_t)^k \to M(A)^k$ be the cellular homotopy equivalences, for $0 \le k \le 2$. Thus, the induced homomorphism $r_{2*} : \pi_2(M(\tilde{A}_t)^2, e_A^0) \to \pi_2(M(A)^2, e_A^0)$ form an isomorphism. So, we can embed the set of generatores $\{g_\alpha^2\}_{\alpha=1}^{13}$ for $\pi_2(M(\tilde{A}_t)^2, e_A^0)$ and the maps $\varphi_\alpha^3 : (S^2, s_0) \to (M(\tilde{A}_t)^2, e^0)$ representing g_α^2 of $\pi_2(M(A)^2, e_A^0)$ by meaning of $r_{2*}(g_\alpha^2)$ and $r_{2*}\varphi_\alpha^3 : (S^2, s_0) \to (M(A)^2, e^0)$, $1 \le \alpha \le 13$. We have the following exact sequence;

$$\begin{tikzcd}
\pi_3(M(A)^3, M(A)^2, e_A^0) \arrow[r, "d"] & \pi_2(M(A)^2, e_A^0) \arrow[r, "i'_*"] \arrow[d, shift right, "i_*"'] & \pi_2(M(A), e_A^0) \neq 0 \\
& \pi_2(M(\tilde{A}_t)^2, e_A^0) \arrow[u, shift right, "i_*^{-1} = r_{2*}"'] &
\end{tikzcd}$$

Let $r_{2*}(g_\alpha^2)$ be the generators of $\pi_2(M(A)^2, e_A^0)$ which are not members of $\ker(i'_*)$ and the number of such generators is 4, the number of all 3-sections that is not NBC bases via the hypersolvable order.

When we retern to $M(\tilde{A}_t)^3$ by the homotopy equivalence *i*, the set of generatores $\{g_\beta^2\}_{\beta=1}^{4}$ generates a subgroup of $\pi_2(M(\tilde{A}_t)^2, e_A^0)$. By the use of Jambu's and Papadima's deformation mathod, we will attach 3-cells $r_{2*}(e_\beta^2)$ by means of φ_β^2, $1 \le \beta \le 4$, to kill off all the 2-holes and make the first non-vanishing higher homotopy group;

$$\pi_2(M(A)^3, e_A^0) \cong \pi_2(M(A), e_A^0) \cong \pi_3(M(\tilde{A}), M(A), e_A^0);$$

vanished, since $\pi_2(M(\tilde{A}_t)^3, e_A^0) \cong \pi_2(M(\tilde{A}_t), e_A^0) \cong 0$. Thus;

$$M(\tilde{A}_t)^3 = M(\tilde{A}_t)^2 \coprod_{\substack{\alpha=1 \\ \varphi_\alpha^2}}^{13} e_\alpha^3$$

$$\approx \left(M(A)^2 \coprod_{\substack{\alpha=1 \\ \varphi_\alpha^2}}^{9} r(e_\alpha^3) \right) \coprod_{\substack{\beta=1 \\ \varphi_\beta^m}}^{4} r(e_\beta^3)$$

$$\approx M(A)^3 \coprod_{\substack{\beta=1\\ \varphi_\beta^2}}^{4} r(e_\beta^3)$$

$$\approx M(A)^3 \coprod_{\substack{\beta=1\\ \varphi_\beta^2}}^{q} S_\beta^3$$

and $\pi_r(M(\widetilde{A}_t), e^0) = \pi_r(M(\widetilde{A}_t)^3, e^0) = \begin{cases} \pi_1(M(A), e^0) & r=1 \\ 0 & r=0,2 \end{cases}.$

As in construction (3.1), we will kill off all the 4th dimensional holes by the same mathod and we get;

$$M(\widetilde{A}_t)^4 = M(\widetilde{A}_t)^3 \coprod_{\substack{\alpha=1\\ \varphi_\alpha^3}}^{4} e_\alpha^4$$

$$\approx \left(M(A)^3 \coprod_{\substack{\beta=1\\ \varphi_\beta^2}}^{4} r(e_\beta^3) \right) \coprod_{\substack{\alpha=1\\ \varphi_\alpha^3}}^{4} e_\alpha^4$$

$$\approx \left(M(A)^3 \coprod_{\substack{\beta=1\\ \varphi_\beta^2}}^{4} r(e_\beta^3) \right) \coprod_{\substack{\alpha=1\\ \varphi_\alpha^3}}^{4} S_\alpha^4$$

and $\pi_r(M(\widetilde{A}_t), e^0) = \pi_r(M(\widetilde{A}_t)^4, e^0) = \begin{cases} \pi_1(M(A), e^0) & r=1 \\ 0 & r\neq 1 \end{cases}.$

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المستخلص:

تضمن هذ االبحث بناء معقدة CW صغرى لمتممة ترتيبة قابلة للحل كليا" مستحث من بنية زمرتها الأساسية كشبه ضرب مباشر تكراري لزمر منتهية التولد لغرض اعطاء تفسير تبولوجي لاختفاء الزمر الهوموتوبية لمتممة ترتيبة قابلة للحل فوقيا" اذا شوهت باستخدام طريقة التشويه لجامبو و باباديما الى عائلة بمتغير واحد من الترتيبات ذات النوع الليفي $\{\widetilde{A}_t\}_{t\in C}$.